LOYOLA COLLEGE (AUTONOMOUS) CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2025**

PMT 2502 – MEASURE THEORY AND INTEGRATION

Date: 26-04-2025 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

SECTION A

Answer ANY FOUR of the following **4 × 10 = 40 Marks**

1. Find the Lebesgue outer measure of an interval.
2. For set E, demonstrate the equivalence of the following statements:
   (i) E is measurable.
   (ii) For all $\epsilon > 0$, there exists an open set, $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$.
   (iii) There exists G, a G_δ-set, $G \supseteq E$ such that $m^*(G - E) = 0$.
   (iv) For all $\epsilon > 0$, there exists a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \epsilon$.
   (v) There exists a F_σ-set F, $F \subseteq E$ such that $m^*(E - F) = 0$.
3. Evaluate $\int_0^1 \frac{x^{1/3}}{1-x} log\left(\frac{1}{x}\right) dx$.
4. If f is Riemann integrable and bounded over the finite interval $[a, b]$, show that the function f is Lebesgue integrable on $[a, b]$ and explain why the following equality holds:
$$\mathcal{R} \int_a^b f dx = \int_a^b f dx.$$
5. Given that μ is a $\sigma - finite\ measure$ on a ring $\mathcal{R}$, analyze the process through which μ can be extended to the $\sigma - ring\ \mathcal{S}(\mathcal{R})$. Additionally, discuss why this extension of μ to $\mathcal{S}(\mathcal{R})$ is unique.
6. Present Minkowski's inequality and discuss its proof.
7. Let $\{f_n\}$ be a sequence of non-negative measurable functions such that $|f_n| < g$, g is an integrable function and $f_n \xrightarrow{m} f$ where f is measurable. Prove the following:
   (i) f is integrable.
   (ii) $\lim \int f_n d\mu = \int f d\mu$.
   (iii) $lim \int |f_n - f|\, d\mu = 0$.
8. State Hahn decomposition for signed measure ν defined on a measurable space $[\![X, S]\!]$ and prove. Is the Hahn decomposition unique? Justify your answer.

SECTION B

Answer ANY THREE of the following **$3 \times 20 = 60$ Marks**

9. Prove that the class of Lebesgue measurable sets is a σ-algebra.

10. Does a non-measurable set exist? Explain your reasoning.

11. State Fatou's Lemma and write the proof. Additionally, provide an example where strict inequality occurs in Fatou's Lemma.

12. Consider a measure μ on a $\sigma - ring\ \mathcal{S}$ and a set $\bar{\mathcal{S}}$ is defined as a collection of sets of the form $E \Delta N$, $E \in \mathcal{S}$ and $N \subseteq M \in \mathcal{S}$ with $\mu(M) = 0$. Explain why $\bar{\mathcal{S}}$ forms a $\sigma - ring$. Consider a set function $\bar{\mu}$ on $\bar{\mathcal{S}}$ by $\bar{\mu}(E \Delta N) = \mu(E)$ and discuss why $\bar{\mu}$ is a complete measure on $\bar{\mathcal{S}}$.

13. (a) Show that every convex function defined on an open interval is continuous.

    (b) Let $[\![X, S, \mu]\!]$ be a measure space with $\mu(X) = 1$ and ψ is convex function on (a, b) where $-\infty < a < b < \infty$. Consider a measurable function f such that $a < f(x) < b$, for all x. Prove the inequality $\psi(\int f d\mu) \leq \int \psi \circ f d\mu$. Also, identify the condition for equality to hold in this inequality and prove it. (8 + 12)

14. Write the statement of the Radon-Nikodym theorem and provide its proof.

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